JOURNAL OF APPROXIMATION THEORY 56, 277–286 (1989)

Multivariate L_p-Error Estimates for Positive Linear Operators via the First-Order τ-Modulus

E. QUAK

Lehrstuhl Mathematik VIII, Universität Dortmund, D-4600 Dortmund 50, West Germany

Communicated by P. L. Butzer

Received June 9, 1986

Let $M(I) := \{f: f$ is a real-valued function that is bounded and measurable on an m-dimensional compact interval $I\}$ and let $L: M(I) \to M(I)$ be a multivariate positive linear operator. The aim of this paper is to give estimates for the approximation error's L_p-norm $\|f - Lf\|_p$ using the so-called averaged modulus of smoothness or τ-modulus of first order.

1. INTRODUCTION

While the univariate averaged modulus of smoothness or τ-modulus has been applied to give estimates for many different purposes (see the monograph [12] for a detailed treatment), the multivariate analog has up to now only been studied in connection with multivariate trigonometric best approximation [5, 6] and the function spaces that are generated by these multivariate τ-moduli [7]. Univariate results concerning estimates for the approximation error of general positive linear operators have been given in [4] via the first-order modulus and in [8, 10] via the second-order modulus. In the following the multivariate τ-modulus will be used for the first time in the context of positive linear operators by establishing a multivariate analogon of Popov's theorem [4] as presented in [9]. Although the theorem stated here is quite similar to the univariate one, the techniques used are completely different because the methods of the univariate proof fail for the multivariate case. Therefore interpolation results of Riesz–Thorin type for the τ-modulus had to be devised instead.

At first, it might be useful to give the detailed definitions of the usual multivariate modulus of smoothness and the multivariate τ-modulus which are both quite straightforward generalisations of the univariate ones.

Let $f: I \to \mathbb{R}$ be a function defined on the compact interval $I \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$.

0021-9045/89 \$3.00

Let $|x|$ denote the maximum norm of a point $x \in \mathbb{R}^m$. For $k \in \mathbb{N}$ and $h \in \mathbb{R}^m$, let $\Delta_h^k f(x)$ denote the kth order difference of step h at the point $x \in I$, i.e.,

$$\Delta_h^k f(x) := \begin{cases} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} f(x+lh) & \text{if } \ x, x+kh \in I \\ 0 & \text{in all other cases.} \end{cases}$$

Let $L_p(I) := \{f\colon f \ p\text{-integrable on } I\}$, $1 \leqq p < \infty$, and $L_\infty(I) := \{f\colon f$ bounded on $I\}$ while $\|\cdot\|_p$ refers to the corresponding norm.

Finally let C stand for a positive real constant, the value of which may be different at each occurrence.

DEFINITION 1.1. Let $f \in L_p(I)$, $1 \leqq p \leqq \infty$, and $k \in \mathbb{N}$, $\delta \in \mathbb{R}^+$. The usual (multivariate) modulus of smoothness of order k and step δ for the function f in L_p-norm is defined as

$$\text{(i)} \quad \omega_k(f; \delta)_p := \sup_{0 < |h| \leqq \delta} \left(\int_I |\Delta_h^k(x)|^p \, dx \right)^{1/p} \text{ for } 1 \leqq p < \infty$$

$$\text{(ii)} \quad \omega_k(f; \delta)_\infty := \sup\{ |\Delta_h^k f(x)| : x, x+kh \in I, \ |h| \leqq \delta \}.$$

Furthermore, let $M(I) := \{f\colon f \text{ bounded and measurable on } I\}$. For these functions the τ-modulus can be given as follows:

DEFINITION 1.2. Let $f \in M(I)$, $k \in \mathbb{N}$, and $\delta \in \mathbb{R}^+$.

(i) The (multivariate) local modulus of smoothness of order k for the function f at the point $x \in I$ and for step δ is defined as

$$\omega_k(f, x; \delta) := \sup\{ |\Delta_h^k f(t)| : t, t+kh \in \Omega_{(1/2)k\delta}(x) \},$$

where

$$\Omega_{(1/2)k\delta}(x) := \{ y \in \mathbb{R}^m \colon |y - x| \leqq \tfrac{1}{2} k\delta \}.$$

(ii) The (multivariate) averaged modulus of smoothness or τ-modulus of order k for the function f and step δ in L_p-norm is given as

$$\tau_k(f; \delta)_p := \|\omega_k(f, \cdot\,; \delta)\|_p, \qquad 1 \leqq p \leqq \infty.$$

Thus the process of taking L_p-norms first and a supremum afterwards (as for the usual modulus of smoothness) is reversed for the definition of the τ-modulus. In the following some properties of the multivariate τ-modulus are stated which will be used later.

Note that by $\alpha = (\alpha_1, ..., \alpha_m)$ an m-dimensional multi-index is denoted with $|\alpha| = \sum_{i=1}^m \alpha_i$ as its norm and $D^\alpha f$ as its corresponding partial derivative of the function f.

LEMMA 1.1. *Let $f, g \in M(I)$, $k \in \mathbb{N}$, $1 \leqq p \leqq \infty$, and $\delta, \delta', \lambda \in \mathbb{R}^+$. Then the multivariate τ-modulus of order k has the following properties*

(i) $\tau_k(f; \delta)_p \leqq \tau_k(f; \delta')_p$ *for* $0 < \delta \leqq \delta'$,

(ii) $\tau_k(f+g; \delta)_p \leqq \tau_k(f; \delta)_p + \tau_k(g; \delta)_p$,

(iii) $\tau_k(f; \delta)_p \leqq 2\tau_{k-1}(f; k/(k-1) \cdot \delta)_p$ *for* $k \geqq 2$,

(iv) $\tau_k(f; \lambda\delta)_p \leqq (2]\lambda[+2)^{k+m} \tau_k(f; \delta)_p$,

(v) $\tau_1(f; \delta)_p \leqq 2 \sum_{|\alpha| \geqq 1} \delta^{|\alpha|} \|D^\alpha f\|_p$, $\alpha_i = 0$ *or* 1, *if* $D^\alpha f \in L_p(I)$ *for all multi-indices* α *with* $|\alpha| \geqq 1$, $\alpha_i = 0$ *or* 1,

(vi) $\omega_k(f; \delta)_p \leqq \tau_k(f; \delta)_p$, $\omega_k(f; \delta)_\infty = \tau_k(f; \delta)_\infty$.

For detailed proofs of these properties see [7] or [8].

2. THE GENERAL RESULT

Without loss of generality only the case $m = 2$ will be considered to simplify notation and to avoid a too complicated use of indices. All techniques that are used in the following text can be easily generalized for the case of three or more variables.

The aim is to prove the following multivariate analogon of Popov's univariate result [4] concerning estimates for the approximation error of positive linear operators via the first-order τ-modulus.

THEOREM 2.1. *Let $I := [a_1, b_1] \times [a_2, b_2]$ and let $L: M(I) \to M(I)$ be a positive linear operator (i.e., a linear operator for which $f \geqq 0$ implies $Lf \geqq 0$ for any $f \in M(I)$) satisfying the following conditions (using the notation $e_{i,j}(t_1, t_2) := t_1^i t_2^j$, $i, j = 0, 1, 2$):*

$$\begin{aligned}
Le_{0,0} &= e_{0,0} \\
Le_{1,0}(x) &= x_1 + \alpha_1(x) \\
Le_{0,1}(x) &= x_2 + \alpha_2(x) \\
Le_{2,0}(x) &= x_1^2 + \beta_1(x) \\
Le_{0,2}(x) &= x_2^2 + \beta_2(x) \qquad \text{for} \quad x = (x_1, x_2) \in I
\end{aligned}$$

and for $M := \max(\|\beta_1 - 2e_{1,0}\alpha_1\|_\infty, \|\beta_2 - 2e_{0,1}\alpha_2\|_\infty)$ let $M \leqq \min(1, (b_1 - a_1)^4, (b_2 - a_2)^4)$.

Then for $f \in M(I)$ the following estimate holds,

$$\|Lf - f\|_p \leqq C\tau_1(f; \sqrt[4]{M})_p, \qquad 1 \leqq p \leqq \infty,$$

where the positive constant C is independent of the operator L and the function f.

While Popov's techniques in the univariate case [4] allow a straightforward proof for any p-norm, $1 \leqq p \leqq \infty$, some crucial estimates cannot be generalised for the multivariate case. Instead, the single cases $p=1$ and $p=\infty$ are proven first, followed by the use of a Riesz–Thorin type interpolation theorem for the case $1<p<\infty$.

3. The Case $p=1$

To prove Theorem 2.1 for the case $p=1$, a lemma is used that is similar to the univariate lemma in [4], giving estimates for the approximation error of truncated power functions σ_c by finding suitable parabolas that are always greater (or always less) than the function σ_c.

Lemma 3.1. *Let the conditions of Theorem 2.1 hold. For any $c=(c_1, c_2) \in I$ define a function $\sigma_c\colon \mathbb{R}^2 \to \mathbb{R}$ by*

$$\sigma_c(t_1, t_2) := (t_1 - c_1)_+^0 (t_2 - c_2)_+^0$$
$$:= \begin{cases} 0 & \text{for} \quad t_1 < c_1 \text{ or } t_2 < c_2 \\ 1 & \text{for} \quad t_1 \geqq c_1 \text{ and } t_2 \geqq c_2. \end{cases}$$

Then for $e_c := L\sigma_c - \sigma_c$ and $x \in I$ the following estimates hold

(i) $|e_c(x)| \leqq M(c_1 - x_1)^{-2}$ *if* $x_1 < c_1$,

(ii) $|e_c(x)| \leqq M(c_2 - x_2)^{-2}$ *if* $x_2 < c_2$,

(iii) $|e_c(x)| \leqq M((c_1 - x_1)^{-2} + (c_2 - x_2)^{-2})$ *if* $x_1 > c_1$ *and* $x_2 > c_2$,

(iv) $|e_c(x)| \leqq 1$ *for all* $x \in I$,

(v) $\|e_c\|_1 \leqq C\sqrt{M}$, *the positive constant C being independent of the operator L.*

Proof of Lemma 3.1. Note first that for the parabolas

$$\psi_{x_1}(t_1, t_2) := (t_1 - x_1)^2 \qquad \text{and} \qquad \psi_{x_2}(t_1, t_2) := (t_2 - x_2)^2 \qquad (x \in I \text{ fixed})$$

we have $M = \sup\{L\psi_{x_1}(x), L\psi_{x_2}(x) : x \in I\}$.

First let $x \in I$ and $a_1 \leqq x_1 < c_1$, $x_2 \in [a_2, b_2]$, which implies $\sigma_c(x) = 0$. Then for the parabola G_{x_1}, defined by $G_{x_1}(t) := (c_1 - x_1)^{-2} \psi_{x_1}(t)$, $t \in I$), it holds that $0 \leqq \sigma_c \leqq G_{x_1}$.

Applying the operator L and using its properties, we obtain after evaluating at the point x: $0 \leqq L\sigma_c(x) \leqq (c_1 - x_1)^{-2} L\psi_{x_1}(x)$ and therefore (i). For $x \in I$, $a_2 \leqq x_2 < c_2$, $x_1 \in [a_1, b_1]$, the parabola $G_{x_2}(t) :=$

$(c_2-x_2)^{-2}\psi_{x_2}(t)$, $t\in I$, can be used similarly to show (ii). For $x\in I$, $c_1<x_1\leqq b_1$, $c_2<x_2\leqq b_2$, we have $\sigma_c(x)=1$. In this case for the parabola $H_x(t):=(c_1-x_1)^{-2}\psi_{x_1}(t)+(c_2-x_2)^{-2}\psi_{x_2}(t)$, $t\in I$, the estimates $1\geqq\sigma_c\geqq 1-H_x$ hold, which in due course give (iii). As $0\leqq\sigma_c\leqq 1$ implies $0\leqq L\sigma_c\leqq 1$, we get (iv). Finally, for $0<h\leqq c_2-a_2$

$$\int_{a_2}^{c_2-h}\int_{a_1}^{b_1}|e_c(x_1,x_2)|\,dx_1\,dx_2\leqq(b_1-a_1)\,Mh^{-1}\qquad\text{follows from (ii).}$$

Similarly, for suitable h we obtain

$$\int_{c_2-h}^{c_2+h}\int_{a_1}^{b_1}|e_c(x_1,x_2)|\,dx_1\,dx_2\leqq 2(b_1-a_1)h$$

and

$$\int_{c_2+h}^{b_2}\int_{c_1-h}^{c_1+h}|e_c(x_1,x_2)|\,dx_1\,dx_2\leqq 2(b_2-a_2)h\qquad\text{from (iv),}$$

$$\int_{c_2+h}^{b_2}\int_{a_1}^{c_1-h}|e_c(x_1,x_2)|\,dx_1\,dx_2\leqq(b_2-a_2)\,Mh^{-1}\qquad\text{from (i)}$$

and

$$\int_{c_2+h}^{b_2}\int_{c_1+h}^{b_1}|e_c(x_1,x_2)|\,dx_1\,dx_2\leqq(b_2-a_2+b_1-a_1)\,Mh^{-1}\qquad\text{from (iii).}$$

Setting $h:=\sqrt{M}$ (and dealing with the integral $\int_{a_2}^{c_2+\sqrt{M}}\int_{a_1}^{b_1}$ instead of

$$\int_{a_2}^{c_2-\sqrt{M}}\int_{a_1}^{b_1}\quad\text{and}\quad\int_{c_2-\sqrt{M}}^{c_2+\sqrt{M}}\int_{a_1}^{b_1}\quad\text{if}\quad\sqrt{M}>c_2-a_2\text{ and so on)}$$

the proposition of (v) can be shown. ∎

Suitable functions Pf, Qf are required such that the error $f-Pf$ is always negative, $f-Qf$ is always positive, and for which $f-Pf$, $f-Qf$, $Pf-L(Pf)$, $Qf-L(Qf)$ can be estimated via the first-order τ-modulus.

LEMMA 3.2. *Let the conditions of Theorem* 2.1 *hold. Then for every* $f\in M(I)$ *there exist functions* Pf, $Qf\in M(I)$, *such that the following estimates hold*:

(i) $Qf\leqq f\leqq Pf$;

(ii) $\|f-Pf\|_1\leqq C\tau_1(f;\sqrt[4]{M})_1$, $\|f-Qf\|_1\leqq C\tau_1(f;\sqrt[4]{M})_1$;

(iii) $\|Pf-L(Pf)\|_1\leqq C\tau_1(f;\sqrt[4]{M})_1$, $\|Qf-L(Qf)\|_1\leqq C\tau_1(f;\sqrt[4]{M})_1$.

The positive constants are independent of the operator L *and the function* f.

Proof. As $M \leqq \min(1, (b_1 - a_1)^4, (b_2 - a_2)^4)$, it is possible to choose $n_1, n_2 \in \mathbb{N}$, $n_1, n_2 \geqq 2$, so that $h_1 := (b_1 - a_1)/n_1 \leqq \sqrt[4]{M} \leqq (b_1 - a_1)/(n_1 - 1) \leqq 2h_1$ and $h_2 := (b_2 - a_2)/n_2 \leqq \sqrt[4]{M} \leqq (b_2 - a_2)/(n_2 - 1) \leqq 2h_2$. Using these fixed steps h_1 and h_2, equidistant knots can be defined by $t_{1,i} := a_1 + ih_1$, $i = 0, 1, ..., n_1$, in the interval $[a_1, b_1]$ and $t_{2,j} := a_2 + jh_2$, $j = 0, 1, ..., n_2$, in the interval $[a_2, b_2]$, and thus a decomposition of the interval I is attained by setting

$$I_{i,j} := [t_{1,i}, t_{1,i+1}) \times [t_{2,j}, t_{2,j+1}) \quad \text{for} \quad i = 0, ..., n_1 - 2, j = 0, ..., n_2 - 2$$

$$I_{n_1-1,j} := [t_{1,n_1-1}, t_{1,n_1}] \times [t_{2,j}, t_{2,j+1}) \quad \text{for} \quad j = 0, ..., n_2 - 2$$

$$I_{i,n_2-1} := [t_{1,i}, t_{1,i+1}) \times [t_{2,n_2-1}, t_{2,n_2}] \quad \text{for} \quad i = 0, ..., n_1 - 2$$

$$I_{n_1-1,n_2-1} := [t_{1,n_1-1}, t_{1,n_1}] \times [t_{2,n_2-1}, t_{2,n_2}].$$

Denoting by $I^*_{i,j}$ the closure of the interval $I_{i,j}$ and setting $\mathfrak{I} := \{0, ..., n_1 - 1\}$, $\mathfrak{J} := \{0, ..., n_2 - 1\}$, we define

$$P_{i,j} := \sup\{f(t) \colon t \in I^*_{i,j}\},$$

$$Q_{i,j} := \inf\{f(t) \colon t \in I^*_{i,j}\} \qquad \text{for} \quad i \in \mathfrak{I}, j \in \mathfrak{J}.$$

Using these definitions the functions Pf, $Qf \in M(I)$ are introduced by setting

$$Pf(t) := P_{i,j}, \qquad Qf(t) := Q_{i,j} \qquad \text{for} \quad t \in I_{i,j}, i \in \mathfrak{I}, j \in \mathfrak{J}.$$

It must be shown now that Pf and Qf have the desired properties. Certainly, (i) holds by definition. Another immediate consequence is the inclusion

$$I^*_{i,j} \subseteq \Omega_{|h|}(t) \qquad \text{for} \quad t \in I_{i,j}, i \in \mathfrak{I}, j \in \mathfrak{J}, \qquad \text{and} \qquad h = (h_1, h_2),$$

from where the inequality $|f(t) - Pf(t)| \leqq \omega_1(f, t; 2|h|)$ for $t \in I$ can be found. Integration gives $\|f - Pf\|_1 \leqq \tau_1(f; 2|h|)_1$ and the proposition of (ii) for Pf follows by taking account of the definition of h and by applying Lemma 1.1(i), (iv). The second part of (ii) can be shown in a similar manner.

We now use the truncated power functions as introduced in Lemma 3.1

to give another representation of the function Pf (writing $\sigma_{i,j}$ instead of $\sigma_{t_{1,i}, t_{2,j}}$),

$$Pf(t) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} (P_{i,j} - P_{i,j-1} - P_{i-1,j} + P_{i-1,j-1})\, \sigma_{i,j}(t)$$

for $t \in I$, setting $P_{i,-1} := P_{-1,j} := 0$. This equality can be verified by straightforward calculation. Application of the operator L yields for $t \in I$

$$\begin{aligned} |L(Pf)(t) - Pf(t)| \leqq \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} &|P_{i,j} - P_{i,j-1} \\ &- P_{i-1,j} + P_{i-1,j-1}|\, |L\sigma_{i,j}(t) - \sigma_{i,j}(t)|. \end{aligned}$$

As $|P_{i,j} - P_{i,j-1} - P_{i-1,j} + P_{i-1,j-1}| \leqq 2\omega_1(f, (t_{1,i}, t_{2,j}); 2|h|)$ for $i > 0$ or $j > 0$ and $\sigma_{0,0} = e_{0,0}$ (which implies $L\sigma_{0,0} = e_{0,0}$), it is found that after integrating the above inequality

$$\|L(Pf) - Pf\|_1 \leqq \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} 2\omega_1(f, (t_{1,i}, t_{2,j}); 2|h|)\, \|L\sigma_{i,j} - \sigma_{i,j}\|_1$$

holds.

Applying Lemma 3.1(v) and recalling the relation between $\sqrt[4]{M}$ and h, the inequality

$$\|L(Pf) - Pf\|_1 \leqq C \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \int_{I_{i,j}} \omega_1(f, (t_{1,i}, t_{2,j}); 2|h|)\, dt$$

is obtained.

As $\Omega_{|h|}(t_{1,i}, t_{2,j}) \subseteq \Omega_{2|h|}(t)$ holds for $t \in I_{i,j}$, we have $\omega_1(f, (t_{1,i}, t_{2,j}); 2|h|) \leqq \omega_1(f, t; 4|h|)$ for $t \in I_{i,j}$ and therefore $\|L(Pf) - Pf\|_1 \leqq C\tau_1(f; 4|h|)_1$. The proposition of (iii) for Pf now follows from Lemma 1.1(i), (iv). The one for Qf can be similarly shown. ∎

Proof of Theorem 2.1 *for the case* $p = 1$. The functions Pf and Qf introduced in Lemma 3.2 are used. As $\|f - Lf\|_1 \leqq \|f - Pf\|_1 + \|Pf - L(Pf)\|_1 + \|L(Pf) - Lf\|_1$, by applying Lemma 3.2(i) the following inequality is obtained:

$$\begin{aligned} \|L(Pf) - Lf\|_1 &\leqq \|L(Pf) - L(Qf)\|_1 \\ &\leqq \|L(Pf) - Pf\|_1 + \|Pf - f\|_1 \\ &\quad + \|f - Qf\|_1 + \|Qf - L(Qf)\|_1. \end{aligned}$$

The proposition of Theorem 2.1 can now be shown by using Lemma 3.2(ii), (iii). ∎

4. The Case $p = \infty$

Proof of Theorem 2.1 *for the case* $p = \infty$. As $\tau_1(f; \delta)_\infty = \omega_1(f; \delta)_\infty$ holds by Lemma 1.1(vi), it is possible to emulate the univariate proof of a theorem by Lupaş and Müller [3].

Let $]\lambda[$ denote the greatest integer that is strictly less than $\lambda \in \mathbb{R}$. Furthermore let $\delta \in \mathbb{R}^+$ and keep $x \in I$ fixed. Then for $t \in I$ and $|t-x| \geqq \delta$ we get $|f(t) - f(x)| \leqq \omega_1(f; \delta^{-1}|t-x|\, \delta)_\infty$ and by a well-known property of the usual modulus of smoothness (see, for example, [11]) $|f(t) - f(x)| \leqq (1 +]\delta^{-1}|t-x|[)\, \omega_1(f; \delta)_\infty$.

As $|t-x| = |t_j - x_j|$ for $j=1$ or $j=2$, we obtain $|f(t) - f(x)| \leqq (1 + \delta^{-2}\psi_{x_j}(t))\, \omega_1(f; \delta)_\infty$. This inequality also holds for $t \in I$ and $|t-x| < \delta$ since in this case $]\delta^{-1}|t-x|[= 0$. Now it is possible to apply the operator L and to evaluate the resulting inequality at the point x, thereby showing $|Lf(x) - f(x)| \leqq (1 + \delta^{-2}M)\, \omega_1(f; \delta)_\infty$ for $x \in I$ and finally that $\|Lf - f\|_\infty \leqq (1 + \delta^{-2}M)\, \omega_1(f; \delta)_\infty$. Setting $\delta := \sqrt{M}$, as $M \leqq 1$, $\|Lf - f\|_\infty \leqq C\omega_1(f; \sqrt{M})_\infty \leqq C\omega_1(f; \sqrt[4]{M})_\infty$ is obtained. ∎

5. The Case $1 < p < \infty$

The proposition of Theorem 2.1 for $1 < p < \infty$ follows from the results for the cases $p = 1$ and $p = \infty$ by making use of the following interpolation theorem of Riesz–Thorin type for positive linear operators.

THEOREM 5.1. *Let* $L\colon M(I) \to M(I)$ *be a positive linear operator, so that for* $\tilde{p}$, p^*, $1 \leqq \tilde{p}, p^* \leqq \infty$, *and a* $\delta \in \mathbb{R}^+$ *the following estimates hold*;

$$\|Lf - f\|_{\tilde{p}} \leqq C\tau_1(f; \delta)_{\tilde{p}},$$

$$\|Lf - f\|_{p^*} \leqq C\tau_1(f; \delta)_{p^*} \qquad \textit{for every} \quad f \in M(I),$$

the positive constants C being independent of the operator L, the function f, and the numbers $\tilde{p}$, p^*.

Then for every $f \in M(I)$ *the following estimate holds true*:

$$\|Lf - f\|_p \leqq C\tau_1(f; \delta)_p,$$

if

$$\frac{1}{p} = \frac{1-\theta}{\tilde{p}} + \frac{\theta}{p^*} \qquad \textit{for} \quad \theta \in \mathbb{R},\ 0 < \theta < 1.$$

Here the positive constant C is independent of the operator L, the function f, and the used norm, too.

For the proof of interpolation theorems for τ-moduli see [1, 2].

6. A Corollary to Theorem 2.1

Using Lemma 1.1(v) it is now possible to give as an immediate consequence of Theorem 2.1

COROLLARY 6.1. *Let the conditions of Theorem* 2.1 *hold and let* $f \in M(I)$, *having partial derivatives* $D^{1,0}f$, $D^{0,1}f$, *and* $D^{1,1}f \in L_p(I)$, $1 \leqq p \leqq \infty$. *Then the following estimate holds,*

$$\|Lf - f\|_p \leqq C(\sqrt[4]{M}\|D^{1,0}f\|_p + \sqrt[4]{M}\|D^{0,1}f\|_p + \sqrt{M}\|D^{1,1}f\|_p),$$

the positive constant C *being independent of the operator* L *and the function* f.

A further investigation of this estimate shows that for $p = \infty$ we have

$$\|Lf - f\|_\infty \leqq C(\sqrt[4]{M}\ \|D^{1,0}f\|_\infty + \sqrt[4]{M}\ \|D^{0,1}f\|_\infty),$$

which means that the term $\sqrt{M}\ \|D^{1,1}f\|_\infty$ can be dropped in this case. On the other hand, a suitable example can be found to show that the term $\sqrt{M}\ \|D^{1,1}f\|_p$ is necessary for the cases $1 \leqq p < 2$ and cannot be dropped from the estimate. See [8] for a detailed consideration. It still remains to be shown what happens for $2 \leqq p < \infty$.

Remark. All techniques used in the preceding treatment can be readily transferred to the case of dimension m, $m \geqq 3$. Without going into details, it should be noted that the result is (see [8]):

Let $L: M(I) \to M(I)$ be a positive linear operator that preserves constants. Then for every $f \in M(I)$ and $1 \leqq p \leqq \infty$ the following estimate holds,

$$\|Lf - f\|_p \leqq C\tau_1(f;\ \sqrt[2m]{M})_p,$$

where $M := \sup\{L\psi_i(x): i = 1, ..., m,\ x \in I\}$ $(\psi_i(t) := (t_i - x_i)^2$ for fixed $x \in I$ and $i = 1, ..., m)$ and $M \leqq \min_{i=1,...,m}(1, (b_i - a_i)^{2m})$.

References

1. L. T. DECHEVSKI, Interpolation of Peetre K-functionals and applications to estimating via integral and average moduli of smoothness, *in* "Proceedings of the Alfred Haar Memorial Conference (Budapest 1985)," North-Holland, Amsterdam, in print.
2. L. T. DECHEVSKI, AND E. QUAK, On interpolation of multivariate moduli of smoothness and its application to error estimates for positive linear operators in L_p, submitted for publication.
3. A. LUPAŞ, AND M. W. MÜLLER, Approximation properties of the M_n-operators. *Aequationes Math.* **5** (1970), 19–37.

4. V. A. Popov, On the quantitative Korovkin theorems in L_p, *C. R. Acad. Bulgare Sci.* **35** (1982), 897–900.
5. V. A. Popov, Onesided approximation of periodic functions of several variables, *C. R. Acad. Bulgare Sci.* **35** (1982), 1639–1642.
6. V. A. Popov, On the onesided approximation of multivariate functions, *in* "Approximation Theory IV" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 657–661, Academic Press, New York, 1983.
7. V. A. Popov and V. Kh. Khristov, Averaged moduli of smoothness for functions of several variables, and the function spaces generated by them, *in* "Proceedings of the Steklov Institutes of Mathematics," Vol. 164, pp. 155–160, "Orthogonal Series and Approximation of Functions," Amer. Math. Soc., Providence, RI, 1983.
8. E. Quak, "Uni- und multivariate L_p-Abschätzungen des Approximationsfehlers positiver linearer Operatoren mit Hilfe des τ-Moduls," Dissertation, Universität Dortmund, 1985.
9. E. Quak, L_p-error estimates for the approximation by positive linear operators, *in* "Approximation Theory V" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), Academic Press, New York, 1986.
10. E. Quak, L_p-error estimates for positive linear operators using the second-order τ-modulus, *Anal. Math.*, in press.
11. F. Schurer, On the order of approximation with generalized Bernstein polynomials, *Indag. Math.* **24** (1962), 484–488.
12. Bl. Sendov, V. A. Popov, "Averaged Moduli of Smoothness," Bulgarian Mathematical Monographs, Vol. 4, Bulgarian Academy of Sciences, Sofia, 1983. [Bulgarian]